Solution exercise set 5

Problem 1

We need to solve

$$\frac{dD}{dt} = k_d C^0_{[OM]} e^{-k_d t} - K_r D \tag{1}$$

supplemented with the boundary condition $D(t = 0) = D_0$. First we solve the homogeneous problem

$$\frac{dD}{dt} = -K_r D. \tag{2}$$

The solution of the homogeneous problem is

$$D(t) = A e^{-K_r t}. \tag{3}$$

Then we consider a constant of integration varying in time $A = A(t)$. This method is called "variation of constants" and can be applied to any inhomogenous problem of the same type as (1), i.e. also for other functions than $k_d C^0_{[OM]} e^{-k_d t}$ or even in the case of K_r depending on time. The method can also be applied to higher order equations.

Inserting $D(t) = A(t)e^{-K_r t}$ in (1) we get

$$\frac{dA}{dt} = k_d C^0_{[OM]} e^{-k_d t} e^{K_r t}. \tag{4}$$

The solution of this equation is $A(t) = \frac{k_d C^0_{[OM]}}{K_r - k_d} e^{(K_r - k_d)t} + A_0$. Inserting this result in (3) we get

$$D(t) = \frac{k_d C^0_{[OM]}}{K_r - k_d} e^{-k_d t} + A_0 e^{-K_r t} \tag{5}$$

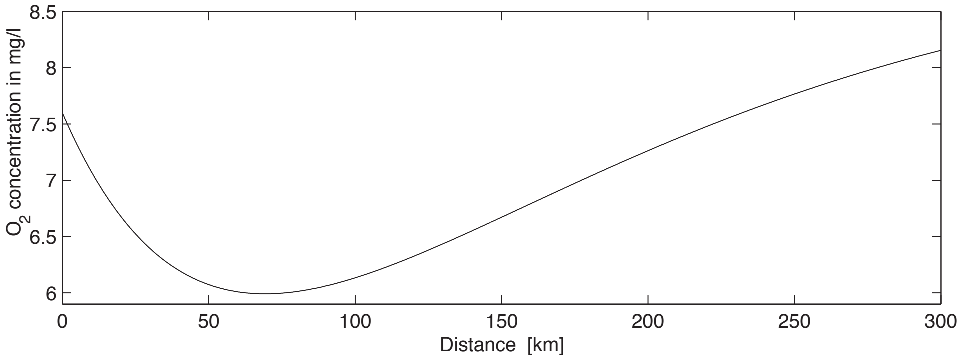
Finally we impose the boundary condition $D(t = 0) = D_0$ and get the final solution

$$D(t) = \frac{k_d C^0_{[OM]}}{K_r - k_d} \left(e^{-k_d t} - e^{-K_r t} \right) + D_0 e^{-K_r t} \tag{6}$$

Problem 2

This problem is solved in details in the book by Socolofsky & Jirka on page 105 (in the book they use the notation L_0 instead of $C^0_{[OM]}$).

Remark that the solution (6) gives us the oxygen deficit over time. To get the position dependence we have to consider that the volume element we follow has a velocity v. Therefore $D(x)$



(with x the distance from the point where the waste is injected) is simply obtained replacing t by $\frac{x}{v}$ in (6)

$$D(x) = \frac{k_d C^0_{[OM]}}{K_r - k_d} \left(e^{-\frac{k_d x}{v}} - e^{-\frac{K_r x}{v}} \right) + D_0 e^{-\frac{K_r x}{v}} \tag{7}$$

The velocity is related to the flow rate

$$v = \frac{Q}{Wh}. \tag{8}$$

Finally remak that since the mass of the waste is given in equivalent oxygen demand we can write $C^0_{[OM]} = \frac{\dot{m}}{Q}$ (a mass per unit of volume).